Resources

- A complete lecture course, including handouts, overheads and papers available from www.mrao.cam.ac.uk/~Clifford

- *Geometric Algebra for Physicists* out in March (C.U.P.)

- David Hestenes’ website modelingnts.la.asu.edu
What is Geometric Algebra?

- Geometric Algebra is a universal Language for physics based on the mathematics of *Clifford Algebra*
- Provides a new product for vectors
- Generalizes complex numbers to arbitrary dimensions
- Treats points, lines, planes, etc. in a single algebra
- Simplifies the treatment of rotations
- Unites Euclidean, affine, projective, spherical, hyperbolic and conformal geometry
Grassmann

German schoolteacher
1809-1877

Published the Lineale
Ausdehnungslehre in 1844

Introduced the outer product

\[ a \wedge b = -b \wedge a \]

Encodes a plane segment
2D Outer Product

- Antisymmetry implies \( a \wedge a = -a \wedge a = 0 \)
- Introduce basis vectors \( e_1, e_2 \)
  \[
a = a_1 e_1 + a_2 e_2 \quad \quad b = b_1 e_1 + b_2 e_2
\]
- Form product
  \[
a \wedge b = a_1 b_2 e_1 \wedge e_2 + a_2 b_1 e_2 \wedge e_1
  = (a_1 b_2 - b_2 a_1) e_1 \wedge e_2
\]
- Returns **area** of the plane + orientation.
- Result is a **bivector**
- Extends (antisymmetry) to arbitrary vectors
Complex Numbers

- Already have a product for vectors in 2D
- Length given by $aa^*$
- Suggests forming

$$ab^* = (a_1 + a_2i)(b_1 - b_2i)$$

$$= (a_1 b_1 + a_2 b_2) - (a_1 b_2 - a_2 b_1)i$$

- Complex multiplication forms the inner and outer products of the underlying vectors!
- Clifford generalised this idea
Hamilton

Introduced his quaternion algebra in 1844

\[ i^2 = j^2 = k^2 = ijk = -1 \]

Generalises complex arithmetic to 3 (4?) dimensions

Very useful for rotations, but confusion over the status of vectors
Quaternions

• Introduce the two quaternion ‘vectors’

\[ a = a_1 i + a_2 j + a_3 k \]
\[ b = b_1 i + b_2 j + b_3 k \]

• Product of these is

\[ ab = c_0 + c \]

• where \( c_0 \) is minus the scalar product and

\[ c = (a_2 b_3 - b_2 a_3) i + (a_3 b_1 - b_3 a_1) j + (a_1 b_2 - b_1 a_2) k \]
W.K. Clifford 1845-1879

Introduced the geometric product

\[ ab = a \cdot b + a \land b \]

Product of two vectors returns the sum of a scalar and a bivector

Think of this sum as like the real and imaginary parts of a complex number.
History

• Foundations of geometric algebra (GA) were laid in the 19th Century
• Key figures: Hamilton, Grassmann, Clifford and Gibbs
• Underused (associated with quaternions)
• Rediscovered by Pauli and Dirac for quantum theory of spin
• Developed by mathematicians (Atiyah etc.) in the 50s and 60s
• Reintroduced to physics in the 70s by David Hestenes
Properties

• Geometric product is **associative and distributive**

\[
a(bc) = (ab)c = abc \\
a(b + c) = ab + ac
\]

• Square of any vector is a **scalar**

\[
(a + b)^2 = a^2 + b^2 + ab + ba
\]

• Define the **inner** (scalar) and **outer** (exterior) products

\[
a \cdot b = \frac{1}{2}(ab + ba) \quad a \wedge b = \frac{1}{2}(ab - ba)
\]
2D Algebra

- Orthonormal basis is 2D
  \[ e_1 \cdot e_1 = e_2 \cdot e_2 = 1 \quad e_1 \cdot e_2 = 0 \]
- Parallel vectors commute
  \[ e_1 e_1 = e_1 \cdot e_1 + e_1 \wedge e_1 = 1 \]
- Orthogonal vectors anticommute since
  \[ e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = -e_2 \wedge e_1 = -e_2 e_1 \]
- Unit bivector has negative square
  \[ (e_1 \wedge e_2)^2 = (e_1 e_2)(e_1 e_2) = e_1 e_2(-e_2 e_1) = -e_1 e_1 = -1 \]
2D Basis

- Build into a basis for the algebra
  \[
  1 \quad \{e_1, e_2\} \quad e_1 \wedge e_2 = I
  \]
  1 scalar \hspace{1cm} 2 vectors \hspace{1cm} 1 bivector
- Even grade objects form **complex numbers**
- Map between vectors and complex numbers

\[
\begin{align*}
  x + Iy &= e_1(xe_1 + ye_2) = e_1x \\
  z^* &= x - Iy = xe_1
\end{align*}
\]
2D Rotations

- In 2D vectors can be rotated using complex phase rotations

\[ v = \exp (i\phi) u \]
\[ u = e_1 x \]
\[ v = e_1 y \]

\[ y = e_1 v = e_1 \exp (I\phi) e_1 x \]

- But

\[ e_1 L e_1 = e_1 (e_1 e_2) e_1 = e_2 e_1 = -I \]

- Rotation

\[ y = \exp (-I\phi) x = x \exp (I\phi) \]
3 Dimensions

• Now introduce a third vector

\{e_1, e_2, e_3\}

• These all anticommute

\[ e_1 e_2 = -e_2 e_1 \text { etc.} \]

• Have 3 bivectors now: \( \{e_1 e_2, e_2 e_3, e_3 e_1\} \)
Bivector Products

• Various new products to form in 3D
• Product of a vector and a bivector
  \[ e_1(e_1e_2) = e_2 \quad e_1(e_2e_3) = e_1e_2e_3 = I \]
• Product of two perpendicular bivectors:
  \[ (e_2e_3)(e_3e_1) = e_2e_3e_3e_1 = e_2e_1 = -e_1e_2 \]
• Set
  \[ i = e_2e_3, \quad j = -e_3e_1, \quad k = e_1e_2 \]
• Recover quaternion relations
  \[ i^2 = j^2 = k^2 = ijk = -1 \]
3D Pseudoscalar

• 3D Pseudoscalar defined by \( I = e_1 e_2 e_3 \)
• Represents a directed volume element
• Has negative square
  \[
  I^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_2 e_3 e_2 e_3 = -1
  \]
• Commutes with all vectors
  \[
  e_1 I = e_1 e_1 e_2 e_3 = -e_1 e_2 e_1 e_3 = e_1 e_2 e_3 e_1 = I e_1
  \]
• Interchanges vectors and planes
  \[
  e_1 I = e_2 e_3 \\
  I e_2 e_3 = -e_1
  \]
3D Basis

Different grades correspond to different geometric objects

Grade 0  |  Grade 1  |  Grade 2  |  Grade 3
Scalar   |  Vector   |  Bivector |  Trivector

1  |  $e_1, e_2, e_3$  |  $e_1 e_2, e_2 e_3, e_3 e_1$  |  $I$

Generators satisfy Pauli relations

$e_i e_j = \delta_{ij} + \epsilon_{ijk}I e_k$

Recover vector cross product

$a \times b = -I a \wedge b$
Reflections

- Build rotations from reflections
- Good example of geometric product – arises in operations

\[ a_\parallel = (a \cdot n)n \]
\[ a_\perp = a - (a \cdot n)n \]

Image of reflection is

\[ b = a_\perp - a_\parallel = a - 2(a \cdot n)n \]
\[ = a - (an + na)n = -nan \]

MIT1 2003
Rotations

- Two rotations form a reflection
  \[ a \rightarrow -m(-nan)m = mnanm \]
- Define the rotor \( R = mn \)
- This is a **geometric** product! Rotations given by
  \[ a \rightarrow RaR^\dagger \]
  \[ R^\dagger = nm \]

- Works in spaces of any dimension or signature
- Works for all grades of multivectors \( A \rightarrow RAR^\dagger \)
- More efficient than matrix multiplication
3D Rotations

- Rotors even grade (scalar + bivector in 3D)
- Normalised: $RR^\dagger = mnnm = 1$
- Reduces d.o.f. from 4 to 3 – enough for a rotation

- In 3D a rotor is a normalised, even element
  \[ R = \alpha + B \quad RR^\dagger = \alpha^2 - B^2 = 1 \]

- Can also write $R = \exp(-B/2)$
- Rotation in plane B with orientation of B
- In terms of an axis $R = \exp(-\theta I n/2)$
Group Manifold

- Rotors are elements of a 4D space, normalised to 1
- They lie on a 3-sphere
- This is the group manifold
- Tangent space is 3D
- Can use Euler angles
  \[ R = \exp(-e_1 e_2 \phi/2) \exp(-e_2 e_3 \theta/2) \exp(-e_1 e_2 \psi/2) \]
- Rotors \( R \) and \( -R \) define the same rotation
- Rotation group manifold is more complicated
Lie Groups

- Every rotor can be written as $R = \pm \exp(-B/2)$
- Rotors form a continuous Lie group
- Bivectors form a Lie algebra under the commutator product
- All finite Lie groups are rotor groups
- All finite Lie algebras are bivector algebras
- (Infinite case not fully clear, yet)
- In conformal case starting point of screw theory (Clifford, 1870s)!
Rotor Interpolation

- How do we interpolate between 2 rotations?
- Form path between rotors

\[
\begin{align*}
R(0) &= R_0 \\
R(1) &= R_1 \\
R(\lambda) &= R_0 \exp(\lambda B)
\end{align*}
\]

- Find B from \( \exp(B) = R_0^\dagger R_1 \)
- This path is \textit{invariant}. If points transformed, path transforms the same way
- Midpoint simply \( R(1/2) = R_0 \exp(-B/2) \)
- Works for \textit{all} Lie groups
Interpolation 2

• For rotors in 3D can do even better!
• View rotors as unit vectors in 4D
• Path is a circle in a plane
• Use simple trig’ to get SLERP

\[ R(\lambda) = \frac{1}{\sin(\theta)} (\sin((1 - \lambda)\theta)R_0 + \sin(\lambda\theta)R_1) \]

• For midpoint add the rotors and normalise!

\[ R(1/2) = \frac{\sin(\theta/2)}{\sin(\theta)} (R_0 + R_1) \]
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\[ a \times b = -Ia \wedge b \]

\[ = b \cdot (Ia) = -a \cdot (Ib) \]
Geometric Algebra 2
Quantum Theory

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Spin

- Stern-Gerlach tells us that electron wavefunction contains two terms
  
  \[ |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \]

- Describe state in terms of a spinor
  
  \[ |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \]

- A 2-state system or qubit
Pauli Matrices

• Operators acting on a spinor must obey angular momentum relations

\[ \hat{l}_i = -i\hbar \epsilon_{ijk} x_j \partial_k, \quad [\hat{l}_i, \hat{l}_j] = i\hbar \epsilon_{ijk} \hat{l}_k \]

• Get spin operators

\[
\begin{align*}
\hat{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\hat{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\hat{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

• These form a **Clifford** algebra

• A **matrix representation** of the geometric algebra of 3D space
Observables

- Want to place Pauli theory in a more geometric framework with \( \vec{\sigma}_k \rightarrow \vec{\sigma}_k \).
- Construct observables
  \[ s_k = \frac{1}{2} \hbar n_k = \langle \psi | \hat{s}_k | \psi \rangle \]
- Belong to a unit vector
- Written in terms of polar coordinates, find parameterisation

\[ |\psi\rangle = \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix} \]
Rotors and Spinors

- From work on Euler angles, encode degrees of freedom in the rotor
  \[ R = e^{-\phi I \sigma_3/2} e^{-\theta I \sigma_2/2} \]

- Represent spinor / qubit as element of the even subalgebra:
  \[ |\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \iff \psi = a^0 + a^k I \sigma_k \]

- Verify that
  \[ \hat{\sigma}_k |\psi\rangle \iff \sigma_k \psi \sigma_3 \quad (k = 1, 2, 3) \]

Keeps result in even algebra
Imaginary Structure

• Can construct imaginary action from

\[ \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \]

• So find that

\[ i |\psi\rangle \leftrightarrow \sigma_1 \sigma_2 \sigma_3 \psi (\sigma_3)^3 = \psi I \sigma_3 \]

• Complex structure controlled by a bivector
• Acts on the right, so commutes with operators applied to the left of the spinor
• Hints at a geometric substructure
• Can always use \( i \) to denote the structure
Inner Products

• The reverse operation in 3D is same as Hermitian conjugation
• Real part of inner product is

\[ \mathbb{R}\langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger | \phi \rangle \]

• Follows that full inner product is

\[ \langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger | \phi \rangle - \langle \psi^\dagger | \phi I \sigma_3 \rangle I \sigma_3 \]

• The projection onto the 1 and \( I \sigma_3 \) components,

\[ \langle A \rangle_q = \frac{1}{2}(A + \sigma_3 A \sigma_3) \]
Observables

- Spin observables become
  \[ \langle \psi | \hat{\sigma}_k | \psi \rangle \leftrightarrow \langle \psi^\dagger \sigma_k \psi \sigma_3 \rangle = \sigma_k \cdot (\psi \sigma_3 \psi^\dagger) \]

- All information contained in the spin vector
  \[ s = \frac{1}{2} \hbar \psi \sigma_3 \psi^\dagger \]

- Now define normalised rotor
  \[ \rho = \psi \psi^\dagger, \quad \psi = \rho^{1/2} R \]

- Operation of forming an observable reduces to
  \[ s = \frac{1}{2} \hbar \rho R \sigma_3 R^\dagger \]

  Same as classical expression
Rotating Spinors

- So have a natural ‘explanation’ for 2-sided construction of quantum observables
- Now look at composite rotations
  \[ R_\theta = \exp(-\hat{\mathcal{B}} \theta / 2), \quad R \sigma_3 R^\dagger \mapsto R_\theta R \sigma_3 R^\dagger R_\theta^\dagger \]

- So rotor transformation law is
  \[ R \mapsto R' = R_\theta R \]

- Take angle through to \(2\pi\)
  \[ R' = e^{-\hat{\mathcal{B}} \pi} R = -R \]

Sign change for fermions
Unitary Transformations

- Spinors can transform under the full unitary group U(2)
- Decomposes into SU(2) and a U(1) term
- SU(2) term becomes a rotor on left
- U(1) term applied on the right
  \[ U(\psi) = R\psi e^{i\phi I\sigma_3} \]
- Separates out the group structure in a helpful way
- Does all generalise to multiparticle setting
Magnetic Field

- Rotor contained in $\psi = \rho^{1/2}R$
- Use this to Simplify equations
- Magnetic field $\hat{H} = -\frac{1}{2}\gamma\hbar B_k \hat{\sigma}_k$
- Schrödinger equation
  $$\frac{d|\psi\rangle}{dt} = \frac{1}{2}\gamma i B_k \hat{\sigma}_k |\psi\rangle$$

- Reduces to simple equation
  $$\dot{\psi} = \frac{1}{2}\gamma B_k I \sigma_k \psi = \frac{1}{2}\gamma I B \psi$$
- Magnitude is constant, so left with rotor equation
  $$\dot{R} = \frac{1}{2}\gamma I B R$$
Density Matrices

• Mixed states are described by a density matrix

• For a pure state this is

\[ \hat{\rho} = |\psi\rangle \langle \psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix} \]

• GA version is

\[ \psi \frac{1}{2} (1 + \sigma_3) \psi^\dagger = \frac{1}{2} (1 + s) \]

• Addition is fine in GA!

• General mixed state from sum

\[ \rho = \frac{1}{2n} (n + s_1 + \cdots + s_n) = \frac{1}{2} (1 + P), \quad |P| \leq 1 \]
Spacetime Algebra

- Basic tool for relativistic physics is the spacetime algebra or STA.

\[ 1 \quad \gamma_\mu \quad \gamma_\mu \gamma_\nu \quad I \gamma_\mu \quad I = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \]

1 scalar 4 vectors 6 bivectors 4 trivectors 1 pseudoscalar

- Generators satisfy

\[ \gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+--), \quad \mu = 0 \ldots 3 \]

- A matrix-free representation of Dirac theory
- Currently used for classical mechanics, scattering, tunnelling, supersymmetry, gravity and quantum information
Relative Space

- Determine 3D space relative for observer with velocity given by timelike vector $\gamma_0$
- Suppose event has position $x$ in natural units
  \[
  t = x \cdot \gamma_0, \quad x = x \wedge \gamma_0
  \]
  \[
  t + x = x \gamma_0
  \]
- The basis elements of relative vector are
  \[
  \sigma_i = \gamma_i \gamma_0
  \]
- Satisfy
  \[
  \sigma_i \cdot \sigma_j = \frac{1}{2} (\gamma_i \gamma_0 \gamma_j \gamma_0 + \gamma_j \gamma_0 \gamma_i \gamma_0)
  = \frac{1}{2} (-\gamma_i \gamma_j - \gamma_j \gamma_i) = \delta_{ij}
  \]
Relative Split

- Split bivectors with $\gamma_0$ to determine relative split

- Relative vectors generate 3D algebra with same volume element

- Relativistic (Dirac) spinors constructed from full 3D algebra
Lorentz Transformation

- Moving observers construct a new coordinate grid
- Both position and time coordinates change

\[ t' = \gamma(t - \beta z), \quad z' = \gamma(z - \beta t), \]
\[ \gamma = (1 - \beta^2)^{-1/2}, \quad \beta = v/c \]

Need to re-express this in terms of \textbf{vector transformations}
Frames and Boosts

- Vector unaffected by coordinate system, so
  \[ x = x^\mu e_\mu = x'^\mu e'_\mu \]

- Frame vectors related by
  \[ e'_0 = \gamma(e_0 + \beta e_3), \quad e'_3 = \gamma(e_3 + \beta e_0) \]

- Introduce the hyperbolic angle
  \[ \tanh(\alpha) = \beta \]

- Transformed vectors now
  \[ e'_0 = \cosh(\alpha) e_0 + \sinh(\alpha) e_3 = \exp(\alpha e_3 e_0) e_0 \]

Exponential of a bivector
Spacetime Rotors

• Define the spacetime rotor

\[ R = e^{\alpha} e_3 e_0 / 2 \]

• A Lorentz transformation can now be written in rotor form

\[ e'_\mu = Re_\mu \tilde{R} \]

• Use the tilde for reverse operation in the STA (dagger is frame-dependent)

• Same rotor description as 3D

• Far superior to 4X4 matrices!
Pure Boosts

- Rotors generate proper orthochronous transformations
- Suppose we want the pure boost from $u$ to $v$

$$v = Lu\tilde{L} = L^2u$$

- Solution is

$$L = \frac{1 + vu}{[2(1 + u \cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2} \frac{v \wedge u}{|v \wedge u|}\right)$$

- Remainder of a general rotor is

$$R = LU, \quad U = \tilde{L}R$$

A 3D rotor
Velocity and Acceleration

• Write arbitrary 4-velocity as

\[ \mathbf{v} = R \gamma_0 \mathbf{\tilde{R}} \]

• Acceleration is

\[ \ddot{\mathbf{v}} = \frac{d}{d\tau} (R \gamma_0 \mathbf{\tilde{R}}) = \dot{R} \gamma_0 \mathbf{\tilde{R}} + R \gamma_0 \dot{\mathbf{\tilde{R}}} \]

• But

\[ \ddot{R} \mathbf{\tilde{R}} = -R \ddot{\mathbf{\tilde{R}}} \]

• So

\[ \ddot{\mathbf{v}} = \ddot{R} \mathbf{\tilde{R}} \mathbf{v} - v \dot{\mathbf{\tilde{R}}} \mathbf{\tilde{R}} = 2(\ddot{\mathbf{R}} \mathbf{\tilde{R}}) \cdot \mathbf{v} \]

Pure bivector

Acceleration bivector
The vector derivative

- Define the vector derivative operator in the standard way

\[ \nabla = \sum_k e^k \frac{\partial}{\partial x^k} \]

- So components of this are directional derivatives
- But now the vector product terms are invertible
- Can construct Green’s functions for \( \nabla \)
- These are Feynman propagators in spacetime
2 Dimensions

• Vector derivative is

\[ \nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} \]

• Now introduce the scalar+psuedoscalar field

\[ \varphi = u + Iv \]

• Find that

\[ \nabla \varphi = \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) e_1 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) e_2 \]

• Same terms that appear in the Cauchy-Riemann equations!
Analytic Functions

- Vector derivative closely related to definition of analytic functions
- Statement that \( \varphi \) is analytic is \( \nabla \varphi = 0 \)
- Cauchy integral formula provides inverse
- This generalises to arbitrary dimensions
- Can construct power series in \( z \) because
  \[
  \nabla z = \nabla (e_1 x) = \nabla (2e_1 \cdot r - xe_1) = 0
  \]
- Lose the commutativity in higher dimensions
- But this does not worry us now!
Spacetime Vector Derivative

• Define spacetime vector derivative

\[ \nabla = \gamma^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \]

• Has a spacetime split of the form

\[ \nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i) \gamma_0 = \partial_t - \sigma_i \partial_i = \partial_t - \nabla \]

• First application - Maxwell equations

\[ \nabla \cdot D = \rho \quad \nabla \cdot B = 0 \]
\[ -\nabla \times E = \frac{\partial}{\partial t} B \quad \nabla \times H = \frac{\partial}{\partial t} D + J \]
Maxwell Equations

• Assume no magnetisation and polarisation effects and revert to natural units
• Maxwell equations become, in GA form

\[ \nabla \cdot E = \rho \quad \nabla \cdot B = 0 \]
\[ \nabla \wedge E = -\partial_t (IB) \quad \nabla \wedge B = I (J + \partial_t E) \]

• Naturally assemble equations for the divergence and (bivector) curl
• Combine using geometric product

\[ \nabla (E + IB) + \partial_t (E + IB) = \rho - J \]
STA Form

- Define the field strength (Faraday bivector)
  \[ F = E + IB \]
- And current
  \[ J\gamma_0 = \rho + J \]
- All 4 Maxwell equations unite into the single equation
  \[ \nabla F = J \]
- Spacetime vector derivative is invertible, can carry out first-order propagator theory
- First-order Green’s function for scattering
Application
Lorentz Force Law

- Non-relativistic form is

\[
\frac{dp}{dt} = q(E + \mathbf{v} \times \mathbf{B})
\]

- Can re-express in relativistic form as

\[
\dot{v} = 2(\dot{\mathbf{R}} \mathbf{R}) \cdot \mathbf{v} = \frac{q}{m} \mathbf{F} \cdot \mathbf{v}
\]

- Simplest form is provided by rotor equation

\[
\dot{\mathbf{R}} = \frac{q}{2m} \mathbf{F} \mathbf{R}
\]
Spin Dynamics

• Suppose that a particle carries a spin vector $s$ along its trajectory

\[ s \cdot v = 0, \quad s = R\gamma_3 \vec{R} \]

• Simplest form of rotor equation then has

\[ \dot{s} = \frac{q}{m} F \cdot s \]

• Non-relativistic limit to this equation is

\[ \dot{s} = \frac{q}{m} s \times B \]

Equation for a particle with $g=2$!
Exercises

• 2 spin-1/2 states are represented by $\phi$ and $\psi$, with accompanying spin vectors

\[ s_1 = \phi \sigma_3 \phi, \quad s_2 = \psi \sigma_3 \psi \]

• Prove that

\[ \frac{\langle \phi | \psi \rangle^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle} = \frac{1}{2} (1 + \cos \theta) \]

\[ s_1 \cdot s_2 = |s_1| |s_2| \cos(\theta) \]

• Given that

\[ L = \frac{1 + vu}{[2(1 + v \cdot u)]^{1/2}} \]

• Prove that

\[ v = L u \tilde{\omega} \]
Geometric Algebra 3
Dirac Theory and
Multiparticle Systems

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Dirac Algebra

- Dirac matrix operators are
  \[
  \tilde{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \quad \tilde{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
  \]

- These act on 4-component Dirac spinors

\[
|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}
\]

- These spinors satisfy a first-order wave equation
  \[
i\tilde{\gamma}^\mu \partial_\mu |\psi\rangle = m |\psi\rangle
\]
STA Form

- Adapt the map for Pauli spinors

\[ |\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \iff \psi = \phi + \eta \sigma_3 \]

- Action of the various operators now

\[ \tilde{\gamma}_\mu |\psi\rangle \iff \gamma_\mu \psi \gamma_0 \]
\[ i |\psi\rangle \iff \psi I \sigma_3 \]
\[ \tilde{\gamma}_5 |\psi\rangle \iff \psi \sigma_3 \]

Imaginary structure still a bivector

Dirac equation

\[ \nabla \psi I \sigma_3 - eA \psi = m \psi \gamma_0 \]
Comments

• Dirac equation based on the spacetime vector derivative
• Same as the Maxwell equation, so similar propagator structure
• Electromagnetic coupling from gauge principal
• Plane wave states have

\[ p\psi = m\psi\gamma_0 \]

A boost plus a rotation
Observables

- Observables are
  1. Gauge invariant
  2. Transform covariantly under Lorentz group

\[ \psi \rightarrow R\psi \]

<table>
<thead>
<tr>
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<th>Standard form</th>
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<th>Frame-free form</th>
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<tr>
<td>Scalar</td>
<td>( \langle \bar{\psi}</td>
<td>\psi \rangle )</td>
<td>( \langle \psi</td>
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<td>Vector</td>
<td>( \langle \bar{\psi}</td>
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<td>Pseudovector</td>
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<td>\gamma_\mu \gamma_5</td>
<td>\psi \rangle )</td>
</tr>
<tr>
<td>Pseudoscalar</td>
<td>( \langle \bar{\psi}</td>
<td>i \gamma_5</td>
<td>\psi \rangle )</td>
</tr>
</tbody>
</table>
Current

- Main observable is the Dirac current
  \[ J = \psi \gamma_0 \bar{\psi} \]
- Satisfies the conservation equation
  \[ \nabla \cdot J = 0 \]
- Understand the observable better by writing
  \[ \psi \bar{\psi} = \rho e^{I\beta} \]
  \[ \psi = \rho^{1/2} e^{I\beta/2} \bar{R} \]
  \[ J = \rho R \gamma_0 \bar{R} \]

A Lorentz transformation
Current 2

• The Dirac current has a wider symmetry group than U(1).
• Take \( \psi \rightarrow \psi \tilde{M} \)
• Require \( M \gamma_0 \tilde{M} = \gamma_0 \)
• Four generators satisfy this requirement
  \[ I \sigma_1, \quad I \sigma_2, \quad I \sigma_3, \quad I \]
• Arbitrary transform
  \[ \psi \rightarrow \psi e^{i b} e^{i \phi} \]

SU(2) \quad U(1)
Streamlines

• The conserved current tells us where the probability density flows
• Makes sense to plot current streamlines
• These are genuine, local observables
• Not the same as following a Bohmian interpretation
• No need to insist that a ‘particle’ actually follows a given streamline
• Tunnelling is a good illustration
Tunnelling
Streamlines

Only front of the packet gets through
Spin Vector

- The Dirac spin observable is
  \[ s = \psi \gamma_3 \tilde{\psi} = \rho R \gamma_0 \tilde{R} \]

- Same structure as used in classical model

- Use a 1D wavepacket to simulate a spin measurement

- Magnetic field simulated by a delta function shock

- Splits the initial packet into 2
Streamlines
Spin Orientation

![Graph of Spin Orientation](image-url)
Multiparticle Space

• Now suppose we want to describe $n$ particles.
• View their trajectories as a path in $4n$ dimensional configuration space
• The vectors generators of this space satisfy

$$\gamma^a_\mu \gamma^b_\nu + \gamma^b_\nu \gamma^a_\mu = \begin{cases} 
0 & a \neq b \\
2\eta_{\mu\nu} & a = b 
\end{cases}$$

• Generators from different spaces anticommute
• These give a means of projecting out individual particle species
N-Particle Bivectors

• Now form the relative bivectors from separate spaces \( \sigma_{i}^{a} = \gamma_{i}^{a} \gamma_{0}^{a} \)

• These satisfy

\[
\sigma_{i}^{1} \sigma_{j}^{2} = \gamma_{i}^{1} \gamma_{0}^{1} \gamma_{j}^{2} \gamma_{0}^{2} = \gamma_{i}^{1} \gamma_{j}^{2} \gamma_{0}^{2} \gamma_{0}^{1} = \gamma_{j}^{2} \gamma_{0}^{2} \gamma_{i}^{1} \gamma_{0}^{1} = \sigma_{j}^{2} \sigma_{i}^{1}
\]

• Bivectors from different spaces commute

• This is the GA implementation of the tensor product

• ‘Explains’ the nature of multiparticle Hilbert space
Complex Structure

- In quantum theory, states all share a single complex structure.
- So in GA, 2 particle quantum states must satisfy

$$\psi(I\sigma_3)^1 = \psi(I\sigma_3)^2$$

$$\psi = -\psi(I\sigma_3)^1 (I\sigma_3)^2 = \psi \frac{1}{2} \left( 1 - (I\sigma_3)^1 (I\sigma_3)^2 \right)$$

- Define the 2 particle correlator

$$E = \frac{1}{2} \left( 1 - (I\sigma_3)^1 (I\sigma_3)^2 \right)$$

$$E^2 = E$$
2-Particle States

• Correlator ensures that 2-particle states have 8 real degrees of freedom

• A 2-particle direct-product state is

\[ |\psi, \phi\rangle \leftrightarrow \psi^{1} \phi^{2} E \]

• Action of imaginary is

\[ i |\psi, \phi\rangle \leftrightarrow \psi^{1} \phi^{2} E (I \sigma_{3})^{1} = \psi^{1} \phi^{2} E (I \sigma_{3})^{2} = \psi^{1} \phi^{2} J \]

• Complex structure now generated by \( J \)

\[ J^{2} = -E \]
Operators

• Action of 2-particle Pauli operators

\[
\begin{align*}
\hat{\sigma}_k \otimes \hat{I} |\psi\rangle & \leftrightarrow - (I\sigma_k)^1 \psi J \\
\hat{\sigma}_k \otimes \hat{\sigma}_l |\psi\rangle & \leftrightarrow - (I\sigma_k)^1 (I\sigma_l)^2 \psi E \\
\hat{I} \otimes \hat{\sigma}_k |\psi\rangle & \leftrightarrow - (I\sigma_k^2) \psi J
\end{align*}
\]

• Inner product

\[
\langle \psi | \phi \rangle \leftrightarrow (\psi, \phi)_q = 2\langle \phi E \tilde{\psi} \rangle - 2\langle \phi J \tilde{\psi} \rangle i
\]

• Examples

\[
\begin{align*}
\langle \psi | \hat{\sigma}_k \otimes \hat{I} |\psi\rangle & \leftrightarrow -2 (I\sigma_k)^1 \cdot (\psi J \tilde{\psi}) \\
\langle \psi | \hat{\sigma}_j \otimes \hat{\sigma}_k |\psi\rangle & \leftrightarrow -2 \left( (I\sigma_j)^1 (I\sigma_k)^2 \right) \cdot (\psi E \tilde{\psi})
\end{align*}
\]
Density Matrices

• A normalised 2 particle density matrix can be expressed as

\[ \hat{\rho} = \frac{1}{4} \left( \hat{I} \otimes \hat{I} + a_k \hat{\sigma}_k \otimes \hat{I} + b_k \hat{I} \otimes \hat{\sigma}_k + c_{jk} \hat{\sigma}_j \otimes \hat{\sigma}_k \right) \]

• So, for example

\[ a_k = \text{tr} \left( \hat{\rho} (\hat{\sigma}_k \otimes \hat{I}) \right) = -2 (I \sigma_k)^{\dagger} \cdot (\psi J \tilde{\psi}) \]

• All of the information in the density matrix is held in the observables

[\psi E \tilde{\psi} \quad \psi J \tilde{\psi}]
Inner products and traces

• Can write the overlap probability as

\[ P(\psi, \phi) = |\langle \psi | \phi \rangle|^2 = \text{tr}(\hat{\rho}_\psi \hat{\rho}_\phi) \]

• So have, for n-particle pure states

\[ P(\psi, \phi) = 2^{n-2} \left( \langle (\psi E \bar{\psi})(\phi E \bar{\phi}) \rangle - \langle (\psi J \bar{\psi})(\phi J \bar{\phi}) \rangle \right) \]

• The partial trace operation corresponds to forming the observables, and throwing out terms

• Clearly see how this is removing information

• For mixed states, can correlate on pseudoscalar
Schmidt Decomposition

• General way to handle 2-particle states is to write as a matrix and perform an SVD

\[
|\psi\rangle = e^{i\chi} \left( \cos\left(\frac{\alpha}{2}\right) e^{\frac{i\tau}{2}} \left( \cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \right) \otimes \left( \cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \right) \right.
\]

\[
+ \sin\left(\frac{\alpha}{2}\right) e^{-\frac{i\tau}{2}} \left( \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \right) \otimes \left( \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \right)
\]
GA Form

- Define the local states / operators

\[ R = \psi(\theta_1, \phi_1) e^{I\sigma_3 \tau/4}, \quad S = \psi(\theta_2, \phi_2) e^{I\sigma_3 \tau/4} \]

- Result of the Schmidt decomposition can now be written

\[ \psi = \rho R^1 S^2 \left( \cos(\alpha/2) + \sin(\alpha/2) I\sigma_2^1 I\sigma_2^2 \right) e^{Jx E} \]

  - Local unitaries
  - Entangling term

- Now have a form which generalises to arbitrary particles
3-Particle States

- The GHZ state is
  \[ \psi = \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle) \]

- GA equivalent
  \[ \psi = \exp\left(\frac{\pi}{4} (I\sigma_2)^1 (I\sigma_2)^2 (I\sigma_2)^3\right) \]

- The W-state is more interesting
  \[ |W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \]

- Goes to
  \[ W = R^1 R^2 R^3 T_{12} T_{13} T_{23} T_{123} (\pi/4) \]
  \[ R_i = e^{\pi/4 I\sigma_2} \]
  \[ T_{12} = \cos(\pi/12) + \sin(\pi/12)(I\sigma_2)^1 (I\sigma_2)^2 \]

MIT3 2003
Singlet State

• An example of an entangled, or non-local, state is the 2-particle singlet state

\[ \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \leftrightarrow \psi = \frac{1}{\sqrt{2}} (I \sigma^{1}_2 - I \sigma^{2}_2) \]

• This satisfies the identity

\[ M^{1}_\epsilon = \tilde{M}^{2}_\epsilon \]

• Gives straightforward proof of invariance

\[ R^{1} R^{2}_\epsilon = R^{1} \tilde{R}^{1}_\epsilon = \epsilon \]

• Observables are

\[ 2\epsilon E\tilde{\epsilon} = 1 + (I \sigma^{1}_k) (I \sigma^{2}_k) \]
Relativistic States

- All of the previous considerations extend immediately to relativistic states
- Can give physical definitions of entanglement for Dirac states
- Some disagreement on these issues in current literature
- Has been suggested that relative observers disagree on entanglement and purity
- More likely that an inappropriate definition has been adopted
Relativistic Singlet

- Can extend the non-relativistic state to one invariant under boosts as well

\[ \eta = \varepsilon (1 - I^1 I^2) \]

- This satisfies

\[ R^1 R^2 \eta = R^1 \tilde{R}^1 \eta = \eta \]

A Lorentz rotor

- This state plays an important role in GA versions of 2-spinor calculus and twistor theory
Multiparticle Dirac Equation

- Relativistic multiparticle quantum theory is a slippery subject!
- Ultimately, most issues sorted by QFT
- Can make some progress, though, e.g. with Pauli principle

\[ I_P = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3, \quad \Gamma_\mu = \frac{1}{\sqrt{2}} (\gamma^1_\mu + \gamma^2_\mu) \]

- Antisymmetrised state constructed via

\[ \psi_-(x) = \psi(x) + I_P \psi(I_P x I_P) I_P \]
Current

- For equal mass particles, basic equation is
  \[ \nabla \psi J = m \psi (\gamma_0^1 + \gamma_0^2) \]
- Get a conserved current in 8D space
  \[ \mathcal{J} = \langle \psi (\gamma_0^1 + \gamma_0^2) \rangle_1 \]
- Pauli principle ensures that
  \[ I_P \mathcal{J} (I_P x I_P) I_P = \mathcal{J}(x) \]
- Ensures that if 2 streamlines ever met, they could never separate
Plots

In both cases the packets pass through each other.
Exercises

• Verify that the overlap probability between 2 states is

\[ P(\psi, \phi) = \frac{\langle (\psi E \psi) (\phi E \phi) \rangle - \langle (\psi J \psi) (\phi J \phi) \rangle}{2 \langle \psi E \psi \rangle \langle \phi E \phi \rangle} \]

• Now suppose that one state is the singlet, and the other is separable. Prove that

\[ P(\psi, \phi) = \langle \frac{1}{2} (1 - P^1 Q^2) \frac{1}{2} (1 + I \sigma^{\dagger}_{k} \sigma_{k}^{2}) \rangle = \frac{1}{4} (1 - \cos \theta) \]

Angle between the spin vectors, or between measuring apparatus
Geometric Algebra 4
2 Final Ideas

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Gravity

- Can construct gravity as a gauge theory
- Predictions fully consistent with general relativity
- General covariance replace by demand that observables are gauge invariant
- Equivalence principal replaced by minimal coupling
- No need for any differential geometry
- Can apply GA ideas directly
The Gauge Fields

- Remove the constraint that coordinate vectors are tied to a frame

\[ e_\mu(x) \rightarrow g_\mu(x) \]

- Generates the ‘metric’ via

\[ g_\mu \cdot g_\nu = g_{\mu \nu}, \quad g_\mu \cdot g_\nu^\nu = \delta_\mu^\nu \]

- Theory invariant under local rotations

\[ g_\mu \rightarrow R g_\mu \tilde{R} \]

- Include gauge fields for Lorentz rotations
- Set of bivector fields
- Field strength is Riemann tensor
Dirac Equation

• The Dirac equation in a gravitational background in

\[ g^\mu \left( \partial_\mu + \frac{1}{2} \Omega_\mu \right) \psi I \sigma_3 = m \psi \gamma_0 \]

Gauge fields

• Observables constructed in precisely the same way
• These are full covariant objects
• All observables are scalar combinations of these
Black Hole

- Gauge fields for a Schwarzschild black hole are extremely simple

\[ g^0 \equiv \gamma^0 \]
\[ g^i \equiv \gamma^i - \left( \frac{2GM}{r} \right)^{1/2} \frac{x^i}{r} \gamma^0 \]

- Metric from this gauge choice is

\[ ds^2 = dt^2 - \left( dr + \left( \frac{2GM}{r} \right)^{1/2} dt \right)^2 - r^2 d\Omega^2 \]

'Flat' Minkowski vectors

Gravitational interaction

Free-fall time
Dirac Equation

• Dirac equation now reduces to

\[ \nabla \psi I \sigma_3 + \gamma_0 \hat{H}_I \psi = m \psi \gamma_0 \]

• All gravitational effects are contained in the scalar Hamiltonian

\[ \hat{H}_I \psi = i \hbar (2GM/r)^{1/2} r^{-3/4} \partial_r (r^{3/4} \psi) \]

Free-fall velocity Radial momentum
The Interaction Hamiltonian

\[ \hat{H}_I \psi = i \hbar (2GM/r)^{1/2} r^{-3/4} \partial_r (r^{3/4} \psi) \]

- All gravitational effects in a *single* term
- This is *gauge dependent*
- In all gauge theories, trick is to
  1. Find a sensible gauge
  2. Ensure that all physical predictions are
gauge invariant
- Hamiltonian is scalar (no spin effects)
- Independent of particle mass
- Independent of \( c \)
Applications

- Can carry out scattering calculations using Feynman diagram techniques
- Construct the gravitational analogue of the Mott scattering formula
- Hamiltonian is non-Hermitian due to delta-function at the origin
- Describes absorption
- Compute a quantum spectrum of bound states
Conformal Geometry

- Totally separate new application of geometric algebra
- Arose from considerations in computer graphics
- Removes deficiencies in the OpenGL use of projective geometry
- Now seen to unite themes in twistor theory, supersymmetry and cosmology
Conformal Points

- Start with the stereographic projection

\[ \hat{r} = \cos(\theta) e_1 + \sin(\theta) e_2 \]

\[ \cos(\theta) = \frac{2x}{1 + x^2} \]

- But this representation involves a unit vector
- Seek a **homogeneous** representation

\[ X = 2xe_1 + (1 - x^2)e_2 + (1 + x^2)\bar{e} \]

\[ X^2 = 0 \]
Distance Geometry

- Distance between points in conformal representation is

\[(x - y)^2 = -\frac{2X \cdot Y}{X \cdot n \cdot Y \cdot n}, \quad n = (e_1 + \bar{e})\]

- Can now use rotors for general Euclidean transformations
- Construct spinors for Euclidean group
- In spacetime, these are twistors
Geometric Primitives

• Take the exterior product of three points to determine a line or circle

\[ A_1 \wedge A_2 \wedge A_3 \wedge X = 0 \]

• Similarly, 4 points describe a sphere

\[ A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge X = 0 \]

• Can intersect lines and spheres
• Computationally very efficient
• Superior to OpenGL
Non-Euclidean Geometry

• Change the distance measure to

\[ d(x, y) = 2\lambda \sin^{-1} \left( \frac{X \cdot Y}{2X \cdot \bar{e} Y \cdot \bar{e}} \right)^{1/2} \]

Spherical geometry

\[ d(x, y) = 2 \sinh^{-1} \left( \frac{X \cdot Y}{2X \cdot e Y \cdot e} \right) \]

Hyperbolic Geometry

• And in spacetime get de Sitter and anti-de Sitter spaces

• All geometries united in a single framework
Hyperbolic Geometry

- Made famous by Escher prints
- Intersect points, lines in exactly the same way
- Only the distance measure changes
Applications

• Besides obvious applications to computational geometry:
• There are many links between relativistic multiparticle quantum states and conformal geometry
• Spinor representation of translations enables constructions of new quantum equations
• Strong links to cosmology and wavefunction of the universe
Resources

• A complete lecture course, including handouts, overheads and papers available from www.mrao.cam.ac.uk/~Clifford
• *Geometric Algebra for Physicists* out in March (C.U.P.)
• David Hestenes’ website modelingnts.la.asu.edu